IV.3. Stationary Markov Processes

February 6, 2008

Recap. Last lecture, we talked about two types of Markov processes: the Poisson process and the Brownian motion process. Both of these processes are lacking another property that can be useful in analyzing stochastic processes, that of stationarity, that we defined some time ago.

Stationarity and some notation. Recall from III.**1**: A stochastic process Y is *stationary* if the moments are not affected by a time shift, i.e.,

$$\langle Y(t_1+\tau)Y(t_2+\tau)\dots Y(t_n+\tau)\rangle = \langle Y(t_1)Y(t_2)\dots Y(t_n)\rangle,$$

for all n, τ, and $t_1, t_2, \dots, t_n$.

A theorem that applies only for Markov processes: A Markov process is stationary if and only if i) $P_1(y,t)$ does not depend on t; and ii) $P_{1|1}(y_2,t_2 \mid y_1,t_1)$ depends only on the difference t_2-t_1. Condition ii) implies that $P_{1|1}(y_2,t_2 \mid y_1,t_1) = P_{1|1}(y_2,t_2+\tau \mid y_1,t_1+\tau)$.

Proof: First suppose that conditions i) and ii) are satisfied. Then

$$\begin{aligned}
\langle Y(t_1)Y(t_2)\dots Y(t_n)\rangle &= \int y_1\dots y_n P_n(y_1,t_1;\dots;y_n,t_n)dy_1\dots dy_n \\
&= \int y_1\dots y_n P_{1|1}(y_n,t_n \mid y_{n-1},t_{n-1})\dots P_{1|1}(y_2,t_2 \mid y_1,t_1)P_1(y_1,t_1)dy_1\dots dy_n \\
&= \int y_1\dots y_n P_{1|1}(y_n,t_n+\tau \mid y_{n-1},t_{n-1}+\tau)\dots P_{1|1}(y_2,t_2+\tau \mid y_1,t_1+\tau)P_1(y_1,t_1+\tau)dy_1\dots dy_n \\
&= \langle Y(t_1+\tau)Y(t_2+\tau)\dots Y(t_n+\tau)\rangle
\end{aligned}$$

Thus all moments are invariant under a time shift τ.

Now suppose that the stationarity condition is satisfied. Specifically, this implies that $\langle Y^n(t)\rangle = \langle Y^n(t+\tau)\rangle$ for all n and τ. Since all the moments of $Y(t)$ and $Y(t+\tau)$ are equal, they must have the same probability distribution. Thus $P_1(y,t) = P_1(y,t+\tau)$ for all τ, and thus it must not depend on τ.

Consider the second moments of the process. If they are invariant under a time shift, it follows that

$$\begin{aligned}
\langle Y(t_1)Y(t_2)\rangle &= \langle Y(t_1+\tau)Y(t_2+\tau)\rangle \\
\int\int y_1y_2P_2(y_1,t_1,y_2,t_2)dy_1dy_2 &= \int\int y_1y_2P_2(y_1,t_1+\tau,y_2,t_2+\tau)dy_1dy_2 \\
\int\int y_1y_2P_{1|1}(y_2,t_2 \mid y_1,t_1)P_1(y_1,t)dy_1dy_2 &= \int\int y_1y_2P_{1|1}(y_2,t_2+\tau \mid y_1,t_1+\tau)P_1(y_1,t)dy_1dy_2.
\end{aligned}$$

These two integrals can only be equal if $P_{1|1}(y_2,t_2 \mid y_1,t_1) = P_{1|1}(y_2,t_2+\tau \mid y_1,t_1+\tau)$, which can only be guaranteed for all τ if $P_{1|1}(y_2,t_2 \mid y_1,t_1)$ is a function of t_2-t_1. This completes the proof.

Therefore, we can define a more compact notation for stationary Markov processes:

$$T_\tau(y_2 \mid y_1) := P_{1|1}(y_2,t_2 \mid y_1,t_1).$$

This allows us to rewrite the Chapman-Kolmogorov Equation

$$P_{1|1}(y_3,t_3 \mid y_1,t_1) = \int P_{1|1}(y_3,t_3 \mid y_2,t_2)P_{1|1}(y_2,t_2 \mid y_1,t_1)dy_2$$

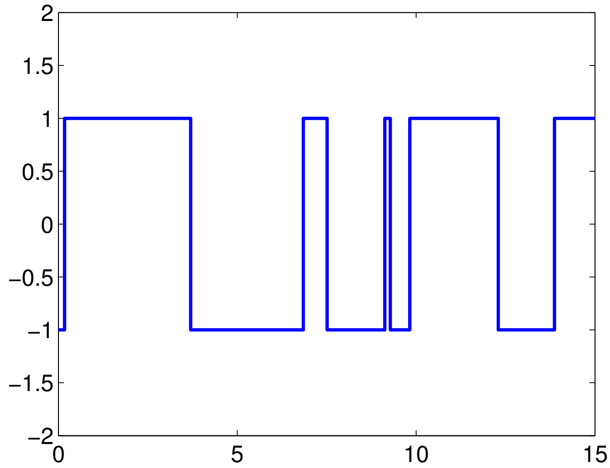

Figure 1: Sample trajectory from a random telegraph process.

as

$$T_{\tau+\tau'}(y_3 \mid y_1) = \int T_\tau(y_3 \mid y_2) T_{\tau'}(y_2 \mid y_1) dy_2$$

where $\tau' := t_3 - t_2$. The Chapman-Kolmogorov Equation only applies when $\tau, \tau' > 0$.

Example. Suppose y_i takes only integer values in the set $\{1, 2, 3, \dots, n\}$. Then we can replace the integral with a sum to get

$$T_{\tau+\tau'}(y_3 \mid y_1) = \sum_{y_2=1}^{n} T_\tau(y_3 \mid y_2) T_{\tau'}(y_2 \mid y_1).$$

This is just the formula for finding each element of a matrix multiplication. So you can think of the Chapman-Kolmogorov equation as being a matrix identity:

$$\mathbf{T}_{\tau+\tau'} = \mathbf{T}_\tau \mathbf{T}_{\tau'}.$$

If the state space is not finite, we can extend this idea from matrices to "integral kernels" in a similar fashion, resulting in the same equation.

Example: Random Telegraph Process. The random telegraph process is defined as a Markov process that takes on only two values: 1 and -1, which it switches between with the rate γ. It can be defined by the equation

$$\frac{\partial}{\partial t} P_1(y, t) = -\gamma P_1(y, t) + \gamma P_1(-y, t).$$

When the process starts at $t = 0$, it is equally likely that the process takes either value, that is

$$P_1(y, 0) = \frac{1}{2}\delta(y - 1) + \frac{1}{2}\delta(y + 1).$$

Goal: To show that the random telegraph process is stationary. We'll need to show that $P_1(y, t)$ does not depend on t and that $P_{1|1}(y_2, y_1 \mid t_2, t_1)$ is a function of $t_2 - t_1$.

How is the number of times that a given trajectory of the process switches between 1 and -1 in a given interval $(t_1, t_2]$? Recall from Monday's lecture, the Poisson process. The process is like a Poisson process except that instead of increasing by 1 each time a new arrival occurs, it switches. We can prove by induction that the distribution of arrivals in any intervals $(t_1, t_2]$ is Poisson.

Base step: The probability there are no switches in an interval $(t', t' + dt']$ is $1 - \gamma dt'$ for small dt'. The probability that there are no switches in $(t_1, t_2]$ is then

$$\Pr(0 \text{ switches in } (t_1, t_2]) = \lim_{dt' \to 0} (1 - \gamma dt')^{\frac{t_2 - t_1}{dt'}} = e^{-\gamma(t_2 - t_1)} = e^{-\gamma(t_2 - t_1)} \frac{(-\gamma(t_2 - t_1))^0}{0!}$$

Induction Step: Assume that the probability of n switches in the interval $(t_1, t_2]$ is $p_n = e^{-\gamma(t_2-t_1)}\frac{(\gamma(t_2-t_1))^n}{n!}$ for $n = 0 \dots N$. Then to find the probability that there are $N+1$ switches in the interval, condition on the time of the 1st switch in the interval, which occurs at time t' with probability $\gamma dt'$. Then there must be 0 switches in the interval $(t_1, t']$ and N switches in the interval $(t', t_2]$. The probability of this is

$$\begin{aligned}
\Pr(\text{N+1 switches in } (t_1, t_2]) &= \int_{t_1}^{t_2} e^{-\gamma(t_2-t')}\frac{(\gamma(t_2-t'))^n}{n!}e^{-\gamma(t'-t_1)}\gamma dt' \\
&= \frac{e^{-\gamma(t_2-t_1)}}{n!}\gamma^{n+1}\int_{t_1}^{t_2}(t_2-t')^n dt' \\
&= \frac{e^{-\gamma(t_2-t_1)}}{n!}\gamma^{n+1}\frac{(t_2-t_1)^{n+1}}{n+1} \\
&= \frac{e^{-\gamma(t_2-t_1)}}{(n+1)!}\left[\gamma(t_2-t_1)\right]^{n+1}
\end{aligned}$$

Now let's find $P_{1|1}(y_2, y_1 \mid t_2, t_1)$. If the trajectory generated by the process switches an even number of times, then $y_1 = y_2$. If it switches an odd number of times, then $y_1 = -y_2$. Therefore

$$P_{1|1}(y_2, t_2 \mid y_1, t_1) = \sum_{n=0,2,4,\dots} e^{-\gamma(t_2-t_1)}\frac{(\gamma(t_2-t_1))^n}{n!}\delta(y_1-y_2) + \sum_{n=1,3,5,\dots} e^{-\gamma(t_2-t_1)}\frac{(\gamma(t_2-t_1))^n}{n!}\delta(y_1+y_2).$$

The power series in the two terms are those of hyperbolic cosine and hyperbolic sine, respectively, so

$$\begin{aligned}
P_{1|1}(y_2, t_2 \mid y_1, t_1) &= e^{-\gamma(t_2-t_1)}\cosh(\gamma(t_2-t_1))\delta(y_1-y_2) + e^{-\gamma(t_2-t_1)}\sinh(\gamma(t_2-t_1))\delta(y_1+y_2) \\
&= e^{-\gamma(t_2-t_1)}\left(\frac{e^{\gamma(t_2-t_1)}+e^{-\gamma(t_2-t_1)}}{2}\right)\delta(y_1-y_2) + e^{-\gamma(t_2-t_1)}\left(\frac{e^{\gamma(t_2-t_1)}-e^{-\gamma(t_2-t_1)}}{2}\right)\delta(y_1+y_2) \\
&= \frac{1}{2}\left(1+e^{-2\gamma(t_2-t_1)}\right)\delta(y_2-y_1) + \frac{1}{2}\left(1-e^{-2\gamma(t_2-t_1)}\right)\delta(y_2+y_1).
\end{aligned}$$

The probability distribution for any time t can be found as follows:

$$\begin{aligned}
P_1(y_2, t) &= P_{1|1}(y_2, t \mid y_1, 0)P_1(y_1, 0) \\
&= \left(\frac{1}{2}\left(1+e^{-2\gamma t_2}\right)\delta(y_2-y_1) + \frac{1}{2}\left(1-e^{-2\gamma t_2}\right)\delta(y_2+y_1)\right)\left(\frac{1}{2}\delta(y_1-1)+\frac{1}{2}\delta(y_1+1)\right) \\
&= \frac{1}{4}\left(1+e^{-2\gamma t_2}\right)\delta(y_2-1) + \frac{1}{4}\left(1-e^{-2\gamma t_2}\right)\delta(y_2-1) + \frac{1}{4}\left(1+e^{-2\gamma t_2}\right)\delta(y_2+1) + \frac{1}{4}\left(1-e^{-2\gamma t_2}\right)\delta(y_2+1) \\
&= \frac{1}{2}\delta(y_2-1) + \frac{1}{2}\delta(y_2+1)
\end{aligned}$$

Is the random telegraph process stationary? Yes, because, i) $P_1(y, t)$ does not depend on t, and ii) $P_{1|1}(y_2, y_1 \mid t_2, t_1)$ is a function of $\tau = t_2 - t_1$. Therefore we can write

$$T_\tau(y_2 \mid y_1) = \frac{1}{2}\left(1+e^{-2\gamma\tau}\right)\delta(y_2-y_1) + \frac{1}{2}\left(1-e^{-2\gamma\tau}\right)\delta(y_2+y_1).$$

Autocorrelation of a stationary process. Since a stationary process has the same probability distribution for all time t, we can always shift the values of the y's by a constant to make the process a zero-mean process. So let's just assume $\langle Y(t)\rangle = 0$. The autocorrelation function is thus:

$$\kappa(t_1, t_1+\tau) = \langle Y(t_1)Y(t_1+\tau)\rangle$$

Since the process is stationary, this doesn't depend on t_1, so we'll denote it by $\kappa(\tau)$. If we know expressions of the transition probability function and the unconditional probability function, we can calculate the autocorrelation function

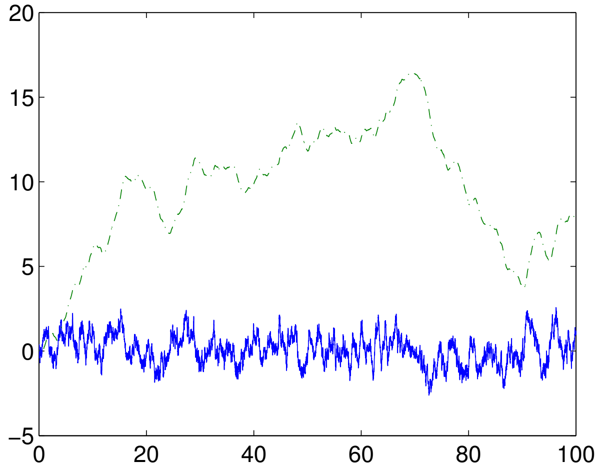

Figure 2: Sample trajectory of the Ornstein-Uhlenbeck process. The dashed line is the integral of the trajectory, which should behave similarly to Brownian motion.

using the formula derived as follows.

$$\begin{aligned}\kappa(\tau) &= \int\int y_1 y_2 P_2(y_1, t_1, y_2, t_1+\tau)dy_1 dy_2 \\ &= \int\int y_1 y_2 P_{1|1}(y_2, t_1+\tau \mid y_1, t_1)P_1(y_1)dy_1 dy_2 \\ &= \int\int y_1 y_2 T_\tau(y_2 \mid y_1)P_1(y_1)dy_1 dy_2.\end{aligned}$$

Example. Autocorrelation of the random telegraph process.

$$\begin{aligned}\kappa(\tau) &= \sum_{y_1\in\{-1,1\}}\sum_{y_2\in\{-1,1\}} y_1 y_2 T_\tau(y_2 \mid y_1)P_1(y_1) \\ &= (1)(1)T_\tau(1 \mid 1)P_1(1) + (-1)(1)T_\tau(-1 \mid 1)P_1(1) + (1)(-1)T_\tau(1 \mid -1)P_1(-1) + (-1)(-1)T_\tau(-1 \mid -1)P_1(-1) \\ &= \frac{1}{2}\frac{1}{2}\left(1+e^{-2\gamma\tau}\right) - \frac{1}{2}\frac{1}{2}\left(1-e^{-2\gamma\tau}\right) - \frac{1}{2}\frac{1}{2}\left(1-e^{-2\gamma\tau}\right) + \frac{1}{2}\frac{1}{2}\left(1+e^{-2\gamma\tau}\right) \\ &= \frac{1}{2}\left(1+e^{-2\gamma\tau}\right) - \frac{1}{2}\left(1-e^{-2\gamma\tau}\right) \\ &= e^{-2\gamma\tau}\end{aligned}$$

The Ornstein-Uhlenbeck Process. The Ornstein-Uhlenbeck process was constructed in order to describe the velocity of a particle in the physical process of Brownian motion. The Ornstein-Uhlenbeck process is a mathematically distinct entity for the Wiener-Levy process that describes the position of a particle in Brownian motion; you can't just integrate and differentiate between the two. It is a stationary Markov process defined by the following equations.

$$\begin{aligned}P_1(y_1) &= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y_1^2} \sim \mathcal{N}(0,1) \\ T_\tau(y_2 \mid y_1) &= \frac{1}{\sqrt{2\pi(1-e^{-2\tau})}}e^{-\frac{(y_2-y_1e^{-\tau})^2}{2(1-e^{-2\tau})}} \sim \mathcal{N}(y_1e^{-\tau}, 1-e^{-2\tau}).\end{aligned}$$

For this process to be properly defined, the functions P_1 and T_τ must satisfy 1) the Chapman-Kolmogorov equation and 2) the consistency condition $\int T_\tau(y_2 \mid y_1)P_1(y_1)dy_1 = P_1(y_2)$. The Ornstein-Uhlenbeck process satisfies condition

2) as shown below:

$$
\begin{aligned}
\int T_\tau(y_2 \mid y_1)P_1(y_1)dy_1 &= \frac{1}{\sqrt{2\pi}}\int \frac{1}{\sqrt{2\pi(1-e^{-2\tau})}} exp\left[-\frac{1}{2}y_1^2 - \frac{(y_2-y_1e^{-\tau})^2}{2(1-e^{-2\tau})}\right]dy_1 \\
&= \frac{1}{\sqrt{2\pi}}\int \frac{1}{\sqrt{2\pi(1-e^{-2\tau})}} exp\left[-\frac{(1-e^{-2\tau})y_1^2 + y_2^2 - 2y_1y_2e^{-\tau} + y_1^2e^{-2\tau}}{2(1-e^{-2\tau})}\right]dy_1 \\
&= \frac{1}{\sqrt{2\pi}}\underbrace{\int \frac{1}{\sqrt{2\pi(1-e^{-2\tau})}} exp\left[-\frac{y_1^2 - 2y_1y_2e^{-\tau} + y_2^2e^{-2\tau}}{2(1-e^{-2\tau})}\right]dy_1}_{=1\text{(Gaussian pdf)}}\, exp\left[-\frac{1}{2}y_2^2\right] \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y_2^2}
\end{aligned}
$$

The Ornstein-Uhlenbeck process also satisfies the Chapman-Kolmogorov equation. The book states that "the reader will have no difficulty in verifying" that these conditions are satisfied. Conceptually, it's not difficult, but it is extremely tedious and skippable. Instead, let's find the autocorrelation function of the process:

$$
\begin{aligned}
\kappa(\tau) &= \int_{y_1}\int_{y_2} y_1P_1(y_1)y_2T_\tau(y_2 \mid y_1)dy_1dy_2 \\
&= \int_{y_1} y_1\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y_1^2}\underbrace{\int_{y_2} y_2\frac{1}{\sqrt{2\pi(1-e^{-2\tau})}}e^{-\frac{(y_2-y_1e^{-\tau})^2}{2(1-e^{-2\tau})}}dy_2}_{=\,y_1e^{-\tau}\text{ (mean of Gaussian pdf)}}dy_1 \\
&= e^{-\tau}\underbrace{\int_{y_1} y_1^2\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y_1^2}dy_1}_{=\,1\text{ (variance of Gaussian pdf)}} \\
&= e^{-\tau}
\end{aligned}
$$

Notice the the autocorrelation function of the Ornstein-Uhlenbeck process is the same form as that of the random telegraph process.

The Ornstein-Uhlenbeck process is interesting because it is essentially the only process that is Gaussian, Markov, and stationary. (Essentially means that processes that are translated in time or space are considered to be the same process, and one pathological process is excluded.) This result is called *Doob's Theorem*. The random telegraph process has only two of these properties: it's Markovian and stationary, but not Gaussian.